Int. J. Solids Structures Vol. 17, pp. 69–81
Pergamon Press Ltd. 1981. Printed in Great Britain

AN AVERAGE THEORY FOR THE DYNAMIC BEHAVIOUR OF A LAMINATED ELASTIC–VISCOPLASTIC MEDIUM UNDER GENERAL LOADING†

J. ABOUDI and Y. BENVENISTE
Department of Solid Mechanics, Materials and Structures, School of Engineering, Tel-Aviv University, Ramat-Aviv, Israel

(Received 23 October 1979; in revised form 7 May 1980)

Abstract—An average theory which models the dynamic behaviour of a bi-laminated composite medium made of elastic–viscoplastic work hardening constituents is presented. The resulting effective theory is represented by a system of nonlinear differential equations for the average stresses, displacements and the plastic work. The theory can be applied to three-dimensional problems under general types of loading.

The theory is applied for the special cases of waves propagating normal to the layering and for waves propagating in a thin composite rod.

INTRODUCTION

It is well known that the determination of the equivalent moduli of a multi-phase medium is very important in the practical applications of those materials.

For multi-phase composites made of linearly elastic constituents the theory of effective moduli is well developed, see for example Hashin[1] and the references cited there. For the special geometry of a periodically bilaminated composite, the effective moduli have been given by Postma[2] and Rytov[3] based on static and dynamic considerations respectively.

For a composite material made of elastic–viscoplastic constituents the effective theory should establish the effective yield law and the equivalent flow rule. It is obvious that the determination of such an average theory will be exceedingly difficult due to the different behaviour of each constituent depending on whether it happens to be in the elastic domain or during a loading or unloading process in the plastic domain.

An elastic–viscoplastic theory for a homogeneous material has been recently developed by Bodner and Partom[4]. In this theory the material is represented by a single set of constitutive equations without a yield criterion, nor loading or unloading conditions. This unified strain-rate dependent theory includes also isotropic hardening.

Based on this unified theory we present in this paper an average model which represents the overall behaviour of a periodically bilaminated composite in which every constituent has an elastic–viscoplastic work hardening behaviour. The derived theory models the overall dynamic behaviour of the composite under the assumption that the lamination thickness is much smaller than the characteristic wavelength so that the microstructure effects of the composite material are ignored. For the special case of a laminated composite with linearly elastic constituents, the presented theory reduces to that derived by Postma[2] in which the composite is represented by a transversely isotropic medium with five equivalent moduli. It is realized that adopting such a unified elastic–viscoplastic law made it possible to model the effective behaviour of the composite, avoiding the above mentioned difficulties.

The derivation is based on the assumption that, for dynamic situations in which the characteristic wavelength is much larger than the lamination thickness, the state of strain and stress is constant in each layer of a representative volume element. The continuity of tractions and displacements at the interfaces of each layer gives the necessary conditions which yield the required constitutive relations between the average stresses and average strains.

The effective theory is expressed in terms of a system of fourteen nonlinear differential equations for the average stress components, average displacements and plastic work. The

†Research sponsored by the Air Force Office of Scientific Research (AFSC) United States Air Force, under Contract No. F49620-79-C-0196. The U.S. Government is authorized to reproduce and distribute reprints for governmental purposes notwithstanding any copyright restrictions.

resulting model appears also in a form of a unified theory which contains neither a yield criterion nor loading or unloading conditions. The obtained theory is completely general and can be used in three-dimensional dynamic situations where the loading conditions and geometry are completely arbitrary, all this generality being of course subjected to the large wavelength restriction.

In Ref. [5], an average theory for an elastic–viscoplastic bilaminated composite was developed for the special case of wave guide type propagation in which the propagating disturbance was generated by a uniformly applied normal loading in the direction of the layering. The method employed in [5] is totally different than the present one and is based on an asymptotic expansion which is truncated properly to yield the average model for this special case. In contrast to the present derivation which is completely general, an important feature of the asymptotic derivation of [5] is that it is extendable to higher order theories taking into account micro-structure effects. As it is expected, for the special case of wave guide type propagation, the present theory yields the same equations as those in [5]. The numerical method for the treatment of the nonlinear differential equations which describe the equivalent model is essentially similar to that given in [5].

The developed theory is applied here to two special situations: (a) Wave propagation normal to the layering of the composite, (b) Uniaxial wave propagation in a thin composite rod.

In case (a) we consider first the case of a uniform loading normal to the layering and we produce the effective stress–strain curve for a given strain-rate. This effective stress–strain curve is compared to those of homogeneous constituents. A laminated slab in which the layers are parallel to its surfaces is then considered. The slab is subjected to a uniform dynamic normal loading on one surface while the other surface is kept rigidly clamped. Here the dynamic response is obtained for uniform normal velocity and stress inputs.

In case (b) a thin rod composed of alternating sections of two materials is considered. The uniaxial constitutive equations corresponding to this geometry are derived from the general theory by imposing the usual thin rod assumptions. The effective uniaxial stress–strain curve for a given strain rate is then obtained and compared to those corresponding to thin rods made of the homogeneous constituents. For a finite thin rod which is impacted at one end by a velocity input, while keeping the other end rigidly clamped, the dynamic response is also obtained.

BASIC EQUATIONS

Consider a periodic array of two alternating isotropic elastic–viscoplastic work-hardening layers. A coordinate system (x_1, x_2, x_3) is defined as seen in Fig. 1 and h_1, h_2 denote the thickness of each layer.

The elastic–viscoplastic work-hardening behaviour of the layers are represented in this paper by the constitutive law proposed by Bodner and Partom[4] which has the important feature of being a unified theory needing no yield criterion nor loading or unloading conditions. In the framework of this theory elastic and inelastic deformations are both present at all stages of loading and unloading.

The constitutive equations of the material $\alpha(\alpha = 1, 2)$ can be described by separating the total strain rate components into elastic (reversible) and plastic (irreversible) strain rates as follows†

$$\dot{\epsilon}_{ij}^{(\alpha)} = \dot{\epsilon}_{ij}^{(e\alpha)} + \dot{\epsilon}_{ij}^{(p\alpha)} \quad i, j = 1, 2, 3 \tag{1}$$

where $\epsilon_{ij}^{(\alpha)} = [u_{i,j}^{(\alpha)} + u_{j,i}^{(\alpha)}]/2$ with $u_i^{(\alpha)}$ being the components of the displacement vector (so that the strains are considered to be infinitesimal), dots represent time derivates and $u_{i,j} = (\partial/\partial x_j)u_i$. The elastic strain rates $\dot{\epsilon}_{ij}^{(e\alpha)}$ are related to the stress rates $\dot{\sigma}_{ij}^{(\alpha)}$ according to the usual Hookes's law

$$\dot{\epsilon}_{ij}^{(e\alpha)} = \dot{\sigma}_{ij}^{(\alpha)}/(2\mu_\alpha) - (\nu_\alpha/E_\alpha)\dot{\sigma}_{kk}^{(\alpha)}\delta_{ij} \tag{2}$$

†Here and in the sequel the subscript or superscript (α) will indicate that quantities belong to either one of the constituents.

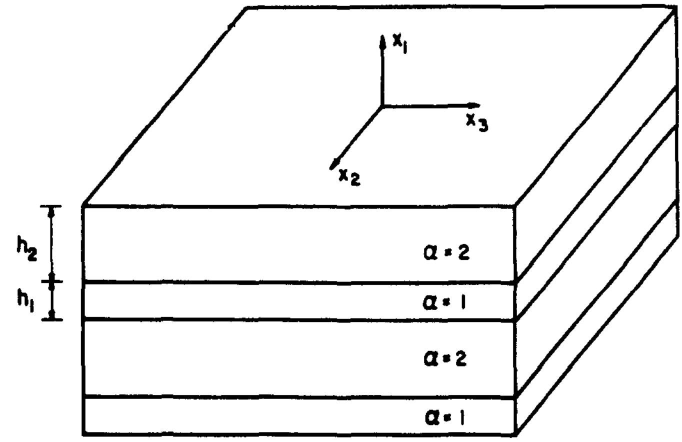

Fig. 1. A laminated medium.

where μ_α, ν_α, E_α are respectively, the rigidity, the Poisson ratio and the Young modulus of the material and δ_{ij} is the Kronecker dealt. The plastic strain rates $\dot\epsilon_{ij}^{(p\alpha)}$ are related to the stresses according to the flow rule.

$$\dot\epsilon_{ij}^{(p\alpha)} = \dot e_{ij}^{(p\alpha)} = \Lambda_\alpha s_{ij}^{(\alpha)} \tag{3}$$

where $s_{ij}^{(\alpha)}$ and $\dot e_{ij}^{(p\alpha)}$ denote the deviators of the stress and plastic rate tensors respectively, i.e. $s_{ij}^{(\alpha)} = \sigma_{ij}^{(\alpha)} - \sigma_{kk}^{(\alpha)}\delta_{ij}/3$ and $\dot e_{ij}^{(p\alpha)} = \dot\epsilon_{ij}^{(p\alpha)} - \dot\epsilon_{kk}^{(p\alpha)}\delta_{ij}/3$. According to (3) the plastic deformations are necessarily incompressible, i.e. $\dot\epsilon_{kk}^{(p\alpha)} = 0$. Equation (3) can be squared to obtain Λ_α.

$$\Lambda_\alpha^2 = D_2^{(p\alpha)}/J_2^{(\alpha)} \tag{4}$$

where

$$D_2^{(p\alpha)} = \dot e_{ij}^{(p\alpha)}\dot e_{ij}^{(p\alpha)}/2, \quad J_2^{(\alpha)} = s_{ij}^{(\alpha)}s_{ij}^{(\alpha)}/2 \tag{5}$$

which are the second invariants of the plastic strain rate deviator and the stress deviator tensors, respectively. Motivated by equations relating dislocation velocities and stresses, Bodner and Partom [4] proposed the relation

$$D_2^{(p\alpha)} = (D_0^{(\alpha)})^2 \exp\{-[z_\alpha^2/(3J_2^{(\alpha)})]^{n_\alpha}(n_\alpha+1)/n_\alpha\} \tag{6}$$

where n_α is related to the steepness of the $D_2^{(p\alpha)} - J_2^{(\alpha)}$ curve, $(D_0^{(\alpha)})^2$ is the limiting value of $D_2^{(p\alpha)}$ for very high stresses and z_α is an internal state variable, referred to as the hardness of the material, which expresses its overall resistance to plastic flow. For isotropic work-hardening, the evolution equation for z_α is taken to depend on the amount of plastic (irreversible) work $w_p^{(\alpha)}$ which has been done on the material from a reference state. Specifically, z_α is assumed to have the form

$$z_\alpha = z_1^{(\alpha)} + (z_0^{(\alpha)} - z_1^{(\alpha)}) \exp[-m_\alpha w_p^{(\alpha)}/z_0^{(\alpha)}] \tag{7}$$

where $z_0^{(\alpha)}$, $z_1^{(\alpha)}$ and m_α are appropriate parameters of the material and the rate of plastic work can be expressed in the form

$$\dot w_p^{(\alpha)} = \sigma_{ij}^{(\alpha)}\dot\epsilon_{ij}^{(p\alpha)} = s_{ij}^{(\alpha)}\ \dot\epsilon_{ij}^{(p\alpha)} = 2\Lambda_\alpha J_2^{(\alpha)}. \tag{8}$$

In (7), $z_0^{(\alpha)}$ is the initial hardness and $z_1^{(\alpha)}$ is the upper limit of z_α (saturation value) since otherwise, $D_2^{(p\alpha)}$ would approach zero for large $w_p^{(\alpha)}$ which leads to fully elastic behaviour at appreciable strains Finally, the stresses fulfil, in the absence of body forces, the usual equations of motion

$$\rho_\alpha \ddot{u}_i^{(\alpha)} = \sigma_{ij,j}^{(\alpha)} \tag{9}$$

where ρ_α is the density of the material.

DEVELOPMENT OF THE AVERAGE THEORY

For dynamic phenomena in which the characteristic wavelength is much larger than the thickness of the lamina, a state of homogeneous stress and strain can be postulated in each layer of a representative volume element of the laminated composite. This assumption is the basis of the derivation of the equivalent moduli of a composite made of linearly elastic layers as presented by Postma[2] and is well justified when the characteristic wavelength is large enough so that no variation of the stresses and strains are felt within one layer. This hypothesis is not valid in regions where strong stress gradients prevail, like in the neighborhood of a concentrated applied load. For the special case of a static problem there's no wavelength considerations and the hypothesis will be justified in regions far away from regions with high stress gradients.

In the considered representative volume element, continuity of the tractions at the plane interfaces (which are perpendicular to the x_1 axis), combined with the constancy of the stresses in each layer yields:

$$\sigma_{1j}^{(1)} = \sigma_{1j}^{(2)} \quad \text{with} \quad j = 1, 2, 3. \tag{10}$$

Under the condition of constant strains in each layer, continuity of displacements at the interfaces is fulfilled only if,

$$\epsilon_{22}^{(1)} = \epsilon_{22}^{(2)}, \ \epsilon_{33}^{(1)} = \epsilon_{33}^{(2)}, \text{ and } \epsilon_{32}^{(1)} = \epsilon_{32}^{(2)} \tag{11}$$

The six conditions given by eqns (10) and (11) will now be used to derive the constitutive relations between the average stresses and average strains over both constituents, relations which in fact characterize the average behaviour of the composite while neglecting microstructure effects.

The average strain $\bar{\epsilon}_{11}$ is defined as

$$\bar{\epsilon}_{11} = N_1 \epsilon_{11}^{(1)} + N_2 \epsilon_{11}^{(2)} \tag{12}$$

where $\epsilon_{11}^{(1)}$ and $\epsilon_{11}^{(2)}$ are the constant strains in each layer, N_1 and N_2 are the volume ratios defined by

$$N_1 = h_1/(h_1 + h_2), \quad N_2 = h_2/(h_1 + h_2) \tag{13}$$

and $\bar{\epsilon}_{11} = (\partial/\partial x_1)\, \bar{u}_1$ where $\bar{u}_1$ is the average displacement in the x_1 direction.

Taking the derivative of (12) with respect to time and using the constitutive relations (1)–(3) for $\dot{\epsilon}_{11}^{(1)}$ and $\dot{\epsilon}_{11}^{(2)}$, we obtain:

$$\begin{aligned}\dot{\bar{\epsilon}}_{11} = \frac{\partial}{\partial x_1} \dot{\bar{u}}_1 &= N_1[(\dot{\bar{\sigma}}_{11}/2\mu_1) - (\nu_1 \dot{\sigma}^{(1)}/E_1)] + N_1 \Lambda_1[\bar{\sigma}_{11} - (\sigma^{(1)}/3)] \\ &+ N_2[(\dot{\bar{\sigma}}_{11}/2\mu_2) - (\nu_2 \dot{\sigma}^{(2)}/E_2)] + N_2 \Lambda_2[\bar{\sigma}_{11} - (\sigma^{(2)}/3)].\end{aligned} \tag{14}$$

$$\sigma^{(\alpha)} = \bar{\sigma}_{11} + \sigma_{22}^{(\alpha)} + \sigma_{33}^{(\alpha)} \tag{15}$$

and

$$\bar{\sigma}_{1j} = \sigma_{1j}^{(\alpha)} \text{ with } j = 1, 2, 3 \tag{16}$$

has been used throughout including in the expressions for Λ_α given by eqns (4)–(8).

Similarly the average strains $\bar{\epsilon}_{12}$ and $\bar{\epsilon}_{13}$ are defined as:

$$\bar{\epsilon}_{1s} = \left(\frac{\partial}{\partial x_s}\bar{u}_1 + \frac{\partial}{\partial x_1}\bar{u}_s\right)/2 = N_1\epsilon_{1s}^{(1)} + N_2\epsilon_{1s}^{(2)} \tag{17}$$

with $s = 2, 3$ and $\bar{u}_2$, $\bar{u}_3$ defining the average displacements in the x_2 and x_3 directions and $\epsilon_{12}^{(\alpha)}$, $\epsilon_{13}^{(\alpha)}$ being the constant strains in each layer. Differentiating (17) with respect to time and using constitutive relations (1)–(3) for $\epsilon_{1s}^{(\alpha)}$ gives:

$$\dot{\bar{\epsilon}}_{1s} = \left(\frac{\partial}{\partial x_s}\dot{\bar{u}}_1 + \frac{\partial}{\partial x_1}\dot{\bar{u}}_s\right)/2 = N_1[(\dot{\bar{\sigma}}_{1s}/2\mu_1\Big) + \Lambda_1\bar{\sigma}_{1s}] + N_2[(\dot{\bar{\sigma}}_{1s}/2\mu_2) + \Lambda_2\bar{\sigma}_{1s}] \tag{18}$$

with $s = 2, 3$. Again the relations (16) have been used in eqn (18), including in Λ_1 and Λ_2.

The average normal strains $\bar{\epsilon}_{22}$ and $\bar{\epsilon}_{33}$ are defined by

$$\bar{\epsilon}_{ss} = \epsilon_{ss}^{(\alpha)} = \frac{\partial}{\partial x_s}\bar{u}_s \quad s = 2, 3\,(\text{no sum}) \tag{19}$$

while the average shear strain $\bar{\epsilon}_{32}$ is given by:

$$\bar{\epsilon}_{32} = \epsilon_{32}^{(\alpha)} = \left(\frac{\partial}{\partial x_2}\bar{u}_3 + \frac{\partial}{\partial x_3}\bar{u}_2\right)/2. \tag{20}$$

Taking the derivative of (19) and (20) with respect to time and using again the constitutive relations (1)–(3) gives

$$\dot{\bar{\epsilon}}_{22} = \frac{\partial}{\partial x_2}\dot{\bar{u}} = (\dot{\sigma}_{22}^{(\alpha)}/2\mu_\alpha) - (\nu_\alpha\dot{\sigma}^{(\alpha)}/E_\alpha) + \Lambda_\alpha[\sigma_{22}^{(\alpha)} - (\sigma^{(\alpha)}/3)] \tag{21}$$

$$\dot{\bar{\epsilon}}_{33} = \frac{\partial}{\partial x_3}\dot{\bar{u}}_3 = (\dot{\sigma}_{33}^{(\alpha)}/2\mu_\alpha) - (\nu_\alpha\dot{\sigma}^{(\alpha)}/E_\alpha) + \Lambda_\alpha[\sigma_{33}^{(\alpha)} - (\sigma^{(\alpha)}/3)] \tag{22}$$

$$\dot{\bar{\epsilon}}_{32} = \left(\frac{\partial}{\partial x_2}\dot{\bar{u}}_3 + \frac{\partial}{\partial x_3}\dot{\bar{u}}_2\right)/2 = (\dot{\sigma}_{32}^{(\alpha)}/2\mu_\alpha) + \Lambda_\alpha\sigma_{32}^{(\alpha)} \tag{23}$$

where $\alpha = 1, 2$ are to be used in (21)–(23). We note now that the average stresses $\bar{\sigma}_{22}$, $\bar{\sigma}_{33}$, and $\bar{\sigma}_{32}$ are given by the definitions

$$\left.\begin{aligned} &\bar{\sigma}_{22} = N_1\sigma_{22}^{(1)} + N_2\sigma_{22}^{(2)}, \quad \bar{\sigma}_{33} = N_1\sigma_{33}^{(1)} + N_2\sigma_{33}^{(2)} \\ &\bar{\sigma}_{32} = N_1\sigma_{32}^{(1)} + N_2\sigma_{32}^{(2)} \end{aligned}\right\} \tag{24}$$

and rewrite again that the plastic work $\dot{w}_p{}^{(\alpha)}$ is given by

$$\dot{w}_p{}^{(\alpha)} = 2\Lambda_\alpha J_2{}^{(\alpha)} \tag{25}$$

where the relation (15) and (16) have been used in (21)–(23) and (25) also.

The system of fourteen equations given by (14), (18) with $s = 2, 3$, (21)–(23), (25) with $\alpha = 1, 2$ and (24) define the constitutive relations of the average theory. They relate in fact the six average stresses $\bar{\sigma}_{ij}$ and their time derivatives $\dot{\bar{\sigma}}_{ij}$ to the average strain rates $\dot{\bar{\epsilon}}_{ij}$ while being coupled to other eight functions $\sigma_{22}^{(\alpha)}$, $\sigma_{33}^{(\alpha)}$, $\sigma_{32}^{(\alpha)}$ and $w_p{}^{(\alpha)}$. Since the starting point was a unified theory without a yield condition nor loading or unloading conditions for the constituents, the obtained average theory for the composite is also unified in the same sense.

The equation of motion for the average model are written formally in terms of the average stresses, displacements and density as follows

$$\bar{\sigma}_{ij,j} = \bar{\rho}\ddot{\bar{u}}_i \tag{26}$$

where $\bar{\rho}$ is taken as $\bar{\rho} = N_1\rho_1 + N_2\rho_2$.

The derived constitutive equations together with the equations of motion (26) form the field equations of the composite represented by an equivalent homogeneous material characterized by the average stresses $\bar{\sigma}_{ij}$ and the average strains $\bar{\epsilon}_{ij}$ (or the average displacements $\bar{u}_i$). The proper boundary conditions of the average model are the same as the classical boundary conditions of a homogeneous medium but will be this time on the average stresses or average displacements (or the velocities for a dynamic case). Equations (14), (18) and (21)–(23) which form part of the constitutive relations can be written now in a compact matrix form as follows:

$$\mathbf{A}\mathbf{X}_{,t} = \mathbf{F} \tag{27}$$

where $X_1 = \bar{\sigma}_{11}$, $X_2 = \bar{\sigma}_{12}$, $X_3 = \bar{\sigma}_{13}$, $X_4 = \sigma_{22}^{(1)}$, $X_5 = \sigma_{22}^{(2)}$, $X_6 = \sigma_{33}^{(1)}$, $X_7 = \sigma_{33}^{(2)}$, $X_8 = \sigma_{23}^{(1)}$, $X_9 = \sigma_{23}^{(2)}$ and $(\ )_{,t}$ denotes differentiation with respect to time. The matrix $\mathbf{A}$ is a nine by nine matrix of constant elements and $\mathbf{F}$ is a column matrix. The elements of both matrices are given in the Appendix.

It will be shown now that for the special case of linearly elastic layers, the system of eqns (27), (24) and (25) furnishes the five equivalent elastic moduli as obtained in [2], which represent the composite as an equivalent homogeneous transversely isotropic medium.

For the case of linearly elastic layers, $\Lambda_\alpha \equiv 0$ and the system of nine coupled equations as given in (27) simplifies into four subsystems. The three subsystems together with the last of (24) furnish in a straightforward manner the constitutive relations between the average shear stresses and average shear strains of the elastic composite. These are:

$$\bar{\sigma}_{12} = 2C_{44}\bar{\epsilon}_{12},\ \bar{\sigma}_{13} = 2C_{44}\bar{\epsilon}_{13},\ \bar{\sigma}_{23} = 2C_{66}\bar{\epsilon}_{23} \tag{28†}$$

where

$$\left.\begin{aligned} C_{44} &= \mu_1\mu_2/(\mu_2N_1 + \mu_1N_2) \\ C_{66} &= \mu_1N_1 + \mu_2N_2\ . \end{aligned}\right\} \tag{29}$$

The remaining fourth subsystem involves five coupled equations given by

$$\mathbf{A}^*\mathbf{Y}_{,t} = \mathbf{F}^* \tag{30}$$

where $Y_1 = \bar{\sigma}_{11}$, $Y_2 = \sigma_{22}^{(1)}$, $Y_3 = \sigma_{22}^{(2)}$, $Y_4 = \sigma_{33}^{(1)}$, $Y_5 = \sigma_{33}^{(2)}$ $F_1^* = \dot{\bar{\epsilon}}_{11}$, $F_2^* = \dot{\bar{\epsilon}}_{22}$, $F_3^* = \dot{\bar{\epsilon}}_{22}$, $F_4^* = \dot{\bar{\epsilon}}_{33}$, $F_5^* = \dot{\bar{\epsilon}}_{33}$ and $\mathbf{A}^*$ is a five by five constant matrix whose elements are given in the Appendix. It is seen that the derivative with respect to time can be eliminated by a simple integration after which the linear algebric equations could be solved. Carrying out the solution and employing the first two of (24), gives the following results:

$$\left.\begin{aligned} \bar{\sigma}_{11} &= C_{33}\bar{\epsilon}_{11} + C_{13}\bar{\epsilon}_{22} + C_{13}\bar{\epsilon}_{33} \\ \bar{\sigma}_{22} &= C_{13}\bar{\epsilon}_{11} + C_{11}\bar{\epsilon}_{22} + C_{12}\bar{\epsilon}_{33} \\ \bar{\sigma}_{33} &= C_{13}\bar{\epsilon}_{11} + C_{12}\bar{\epsilon}_{22} + C_{11}\bar{\epsilon}_{33} \end{aligned}\right\} \tag{31}$$

where

$$\left.\begin{aligned} C_{11} &= [(\lambda_1 + 2\mu_1)(\lambda_2 + 2\mu_2) + 4N_1N_2(\mu_1 - \mu_2)(\lambda_1 + \mu_1 - \lambda_2 - \mu_2)/D \\ C_{13} &= [\lambda_1N_1(\lambda_2 + 2\mu_2) + \lambda_2N_2(\lambda_1 + 2\mu_1)]/D \\ C_{12} &= [\lambda_1\lambda_2 + 2(\lambda_1N_1 + \lambda_2N_2)(\mu_2N_1 + \mu_1N_2)]/D \\ C_{33} &= [(\lambda_1 + 2\mu_1)(\lambda_2 + 2\mu_2)]/D \end{aligned}\right\} \tag{32}$$

†We note that in our case x_1 is perpendicular to the layering and it corresponds to z in [2].

and D is given by

$$D = N_1(\lambda_2 + 2\mu_2) + N_2(\lambda_1 + 2\mu_1).$$

It is a simple matter to check that $2C_{66} = (C_{11} - C_{13})$. The five independent constants C_{44}, C_{11}, C_{12}, C_{13} and C_{33} are in fact the five equivalent moduli of the equivalent homogeneous transversely isotropic medium as given in [2].

The constitutive relations as defined by eqns (27), (24) and (25) can be solved for given strain rates $\dot{\epsilon}_{ij}$ by using the fourth order Runge-Kutta method. As to the solution of a dynamic problem which will involve the dynamic eqns (26) as well as the constitutive relations, they can be solved by a finite difference procedure which is basically a generalization of the procedure described in our previous paper[5] which considers a one-dimensional problem of wave propagation.

APPLICATIONS

The formulated equivalent model is applied to the following situations:

(a) Wave propagation normal to the layering;

(b) Wave propagation in a thin composite rod.

Results are given for a laminated medium made of titanium ($\alpha = 1$) and copper ($\alpha = 2$). The appropriate parameters of these materials are given by [4, 6]:

$$\nu_1 = 0.34, \quad \mu_1 = 0.44 \times 10^{11}\,\text{N/m}^2, \quad D_0^{(1)} = 10^4\,\text{sec}^{-1},$$
$$n_1 = 1, \quad z_0^{(1)} = 1.15 \times 10^9\,\text{N/m}^2, \quad z_1^{(1)} = 1.4 \times 10^9\,\text{N/m}^2,$$
$$m_1 = 100, \quad \rho_1 = 4.87 \times 10^3\,\text{kg/m}^3.$$
$$\nu_2 = 0.33, \quad \mu_2 = 0.45 \times 10^{11}\,\text{N/m}^2, \quad D_0^{(2)} = 10^4\,\text{sec}^{-1}$$
$$n_2 = 7.5, \quad z_0^{(2)} = 0.63 \times 10^8\,\text{N/m}^2, \quad z_1^{(2)} = 2.5 \times 10^8\,\text{N/m}^2$$
$$m_2 = 8.19, \quad \rho_2 = 8.96 \times 10^3\,\text{kg/m}^3.$$

The reinforcement ratios are chosen to be $N_1 = N_2 = 1/2$.

(a) *Wave propagation normal to the layering*

In this section we consider the case of one-dimensional wave propagation in the x_1-direction. First, the effective stress-strain curves are obtained for a laminated medium subjected to a uniform normal loading in the x_1-direction and then the dynamic problem of a laminated slab with the layering parallel to its surfaces is considered.

For the case of a uniform loading in the x_1-direction, $(\partial/\partial x_2) = 0$, $(\partial/\partial x_3) = 0$ and $\bar{u}_2 = \bar{u}_3 = 0$. The constitutive equations (27) reduce in this case to the system of five equations:

$$\mathbf{B}\mathbf{V}_{,t} = \mathbf{Q} \tag{33}$$

where $V_1 = \bar{\sigma}_{11}$, $V_2 = \sigma_{22}^{(1)}$, $V_3 = \sigma_{22}^{(2)}$, $V_4 = \sigma_{33}^{(1)}$, $V_5 = \sigma_{33}^{(2)}$. The elements of the five by five by square matrix $\mathbf{B}$ and column matrix $\mathbf{Q}$ are given in the Appendix. The equations for the plastic work (25) still apply, but this time in the expressions for Λ_α and J_α the following conditions need to be substituted:

$$\sigma_{23}^{(\alpha)} = \bar{\sigma}_{13} = \bar{\sigma}_{12} = 0. \tag{34}$$

The equations of motion (26) in this special case reduce to:

$$\frac{\partial}{\partial x_1}\bar{\sigma}_{11} = \bar{\rho}\ddot{\bar{u}}_1.$$

The system of the differential equations (33) together with the two equations of plastic work (25) in which conditions (34) are substituted can now be employed to construct the effective stress strain curves of the laminated medium, that is, the plot of the average stress $\bar{\sigma}_{11}$ vs the average strain $\bar{\epsilon}_{11} = (\partial/\partial x_1)\ \bar{u}_1$ for a given value of the strain rate $\dot{\bar{\epsilon}}_{11}$. This is achieved by choosing $\dot{\bar{\epsilon}}_{11} = \epsilon_0$ so that $\bar{\epsilon}_{11} = \epsilon_0 t$ and substituting it in the system (33) and (25). The system of the differential equations is then solved numerically using the fourth order Runge-Kutta method as was done in [5].

In Fig. 2 we present the effective stress-strain curve for a laminated medium made of titanium and copper. In the same figure we give also for comparison the stress-strain curves for the homogeneous media made of titanium only and copper only. The prescribed strain rates for all the curves is taken to be $\dot{\epsilon}_0 = 10^{-2}\ \text{sec}^{-1}$. All the stress values in the figure are normalized with respect to the material constant $\lambda + 2\mu$ of titanium which is given by $(\lambda + 2\mu)_{Ti} = 1.82 \times 10^{11}\ \text{N/m}^2$. The nondimensional stress is therefore given by: $\bar{S}_{11} = \bar{\sigma}_{11}/(\lambda + 2\mu)_{Ti}$.† The change of slope from the elastic region to the plastic region is clearly seen in all the curves. The effective stress-strain curve of the composite falls between the curves of the homogeneous cases as expected. In all the curves the point of yielding can be clearly determined. It should be noted that the present theory provides, in particular, the effective yield point of the two-phase laminated medium.

We consider now a laminated slab of thickness H with the layering parallel to its surfaces and occupying the domain $0 \le x_1 \le H$, $-\infty < x_2 < +\infty$, $-\infty < x_3 < +\infty$. The surface $x_1 = H$ is rigidly clamped and the surface $x_1 = 0$ is subjected to a normal velocity or stress input. For the velocity loading input we chose the following function:

$$\dot{\bar{u}}(0, t) = f(t) = \begin{cases} f_0 \sin \pi t/2\tau_m & 0 \le t \le \tau_m \\ f_0 & t \ge \tau_m \end{cases} \tag{35}$$

with $f_0/(l/T) = 0.01$, $\tau_m/T = 0.5$ and the non-dimensionalization factor l/T is chosen with $T = (1/D_0)_{Ti}$, $l = (1/D_0)_{Ti}[(\lambda + 2\mu)_{Ti}/\bar{\rho}]^{1/2}$ The slab width is taken as $H = l$.

In Fig. 3 the resulting average nondimensional velocity $\bar{U}_{,\tau} = \dot{\bar{u}}_1/(l/T)$ is given vs the nondimensional time $\tau = t/T$. Results are produced at the nondimensional observation points $\xi = x_1/l = 0.3$ and $\xi = x_1/l = 0.7$ within the slab. This velocity field of the elastic-viscoplastic composite is compared with that of laminated slab of titanium and copper but assuming that their behaviour is perfectly elastic. It is noted that the basic effect of the viscoplastic mechanism is in the spreading and attenuation of the propagating pulse.

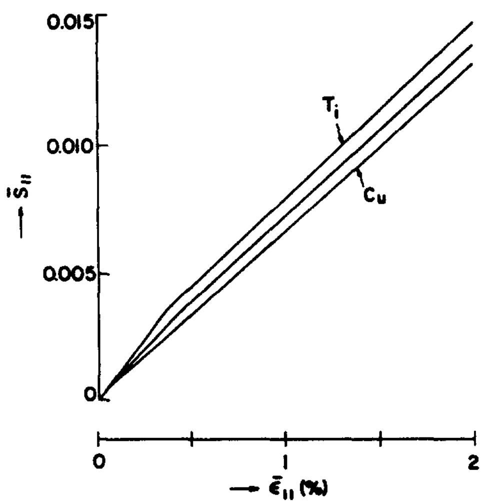

Fig. 2. Effective stress-strain curve for an elastic-viscoplastic laminated medium made of titanium and copper, and the stress-strain curves for a homogeneous medium made of (a) titanium (Ti), (b) copper (Cu). The strain rate in all cases is $10^{-2}\ \text{sec}^{-1}$.

†The subscript Ti denotes here and the sequel that the relevant parameters are those of titanium.

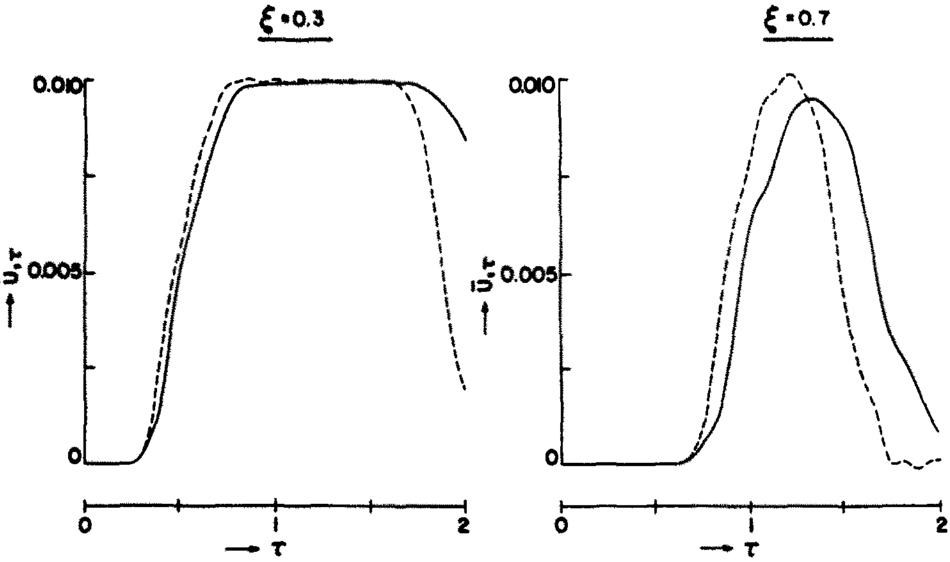

Fig. 3. The average velocity vs time at $\xi = 0.3$ and 0.7 in an elastic-viscoplastic laminated slab (solid lines) and in a perfectly elastic laminated slab (dashed lines). The velocity input is given by (35).

A stress input at $x_1 = 0$ is now considered and is taken as follows:

$$\sigma_{11}(0, t) = g(t) \tag{36}$$

where $g(t) = [(\lambda + 2\mu)_{\mathrm{Ti}}]\, f(t)/(l/T)$ with $f(t)$ being defined in (35).

In Fig. 4 the average stress field $\bar{S}_{11} = \bar{\sigma}_{11}/(\lambda + 2\mu)_{\mathrm{Ti}}$ is given vs the non-dimensional time $\tau = t/T$ at the same observation points as those of Fig. 3. This average stress field of the equivalent model is also compared in the figure with the corresponding one in a perfectly elastic laminated slab.

We should note here that the one-dimensional problem of wave propagation in the direction of the layering can also be investigated in the framework of the constructed general model. The same treatment presented in this section for propagating waves normal to the layering can be carried out in this case too. That is, the relevant constitutive equations can be obtained, the average stress–strain curve can be constructed and the dynamic problem of a laminated slab with the layers perpendicular to its surfaces and occupying the domain $0 \le x_2 \le H$, $-\infty < x_1 < \infty$,

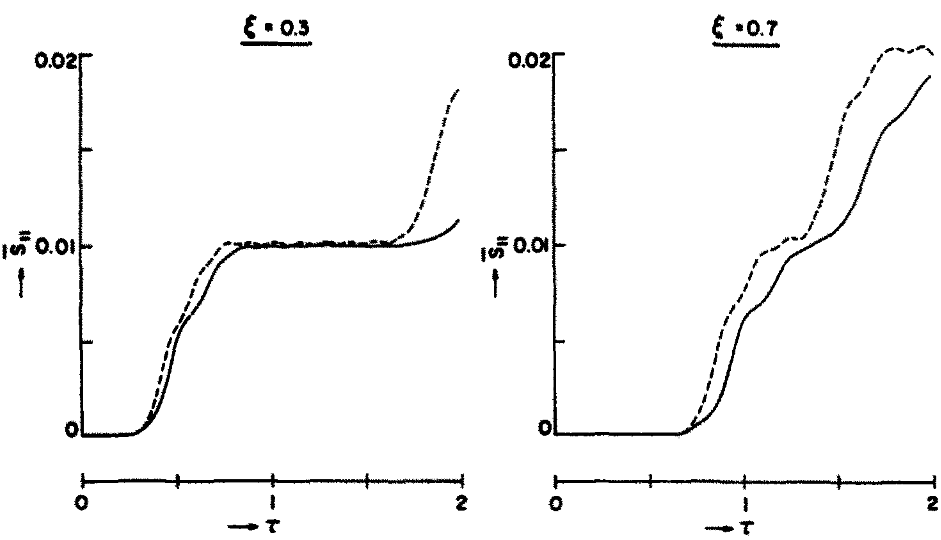

Fig. 4. The average stress vs time at $\xi = 0.3$ and 0.7 in an elastic-viscoplastic laminated slab (solid lines) and in a perfectly elastic laminated slab (dashed lines). The stress input is given by (36).

$-\infty < x_3 < +\infty$, can be investigated. In fact, this problem has been treated in the previous paper by the authors[5] where the average model was derived by a totally different method. The procedure used in [5] was based on asymptotic expansions and the obtained average model was limited to wave-guide type propagation only. While the governing equations of the equivalent model given in [5] are obtained as a special case of the general theory constructed in this paper, the asymptotic method employed there has the important advantage of being extendable to the construction of higher order models including microstructure effects.

(b) *Wave propagation in a composite thin rod*

We consider here a composite thin rod made of alternating sections of two elasto-viscoplastic work hardening materials. The x_1-coordinate is taken parallel to the rod axis. Since the rod is thin we can apply the well known assumptions of the elementary thin rod theory, that is,

$$\bar{\sigma}_{11} \neq 0,\ \bar{\sigma}_{12} = \bar{\sigma}_{13} = \bar{\sigma}_{22} = \bar{\sigma}_{33} = \bar{\sigma}_{22} = 0. \tag{37}$$

Conditions (37) will now be used in conjunction with eqns (14), (18), (21)–(23) to derive the governing constitutive relations of the described composite thin rod. The last three equalities in eqn (37) imply:

$$\left.\begin{aligned} &\sigma_{22}^{(1)} = -(N_2/N_1)\sigma_{22}^{(2)},\ \sigma_{33}^{(1)} = -(N_2/N_1)\sigma_{33}^{(2)}, \\ &\sigma_{23}^{(1)} = -(N_2/N_1)\sigma_{23}^{(2)}. \end{aligned}\right\} \tag{38}$$

Using eqns (38) in (14) gives

$$\begin{aligned} \frac{\partial}{\partial x_1}\dot{u}_1 = {} & N_1\{(\dot{\bar{\sigma}}_{11}/2\mu_1) - (\nu_1/E_1)[\dot{\bar{\sigma}}_{11} - (N_2/N_1)\dot{\sigma}_{22}^{(2)} - (N_2/N_1)\dot{\sigma}_{33}^{(2)}]\} \\ & + N_1\Lambda_1[\bar{\sigma}_{11} - (\sigma^{(1)}/3)] \\ & + N_2\{(\dot{\bar{\sigma}}_{11}/2\mu_2) - (\nu_2/E_2)[\dot{\bar{\sigma}}_{11} + \dot{\sigma}_{22}^{(2)} + \dot{\sigma}_{33}^{(2)}]\} \\ & + N_2\Lambda_2[\bar{\sigma}_{11} - (\sigma^{(2)}/3)] \end{aligned} \tag{39}$$

where first two relations of (38) need now to be used in the definition of $\sigma^{(1)}$ given in (15), and also in the expressions for Λ_α.

Using again eqns (38), equating the two eqns of (21) with $\alpha = 1, 2$ to each other, and doing the same for the relations in (22) and (23), furnishes the following three equations

$$\begin{aligned} & (\dot{\sigma}_{22}^{(2)}/2\mu_1)(-N_2/N_1) - (\nu_1/E_1)[\dot{\bar{\sigma}}_{11} - (\dot{\sigma}_{22}^{(2)}N_2/N_1) - (\dot{\sigma}_{33}^{(2)}N_2/N_1)] \\ & + \Lambda_1[(-\sigma_{22}^{(2)}N_2/N_1) - (\sigma^{(1)}/3)] \\ & = (\dot{\sigma}_{22}^{(2)}/2\mu_2) - (\nu_2/E_2)[\dot{\bar{\sigma}}_{11} + \dot{\sigma}_{22}^{(2)} + \dot{\sigma}_{33}^{(2)}] + \Lambda_2[\sigma_{22}^{(2)} - (\sigma^{(2)}/3)]. \end{aligned} \tag{40}$$

$$\begin{aligned} & (\dot{\sigma}_{33}^{(2)}/2\mu_1)(-N_2/N_1) - (\nu_1/E_1)[\dot{\bar{\sigma}}_{11} - (\dot{\sigma}_{22}^{(2)}N_2/N_1) - (\dot{\sigma}_{33}^{(2)}N_2/N_1)] \\ & + \Lambda_1[(-\sigma_{33}^{(2)}N_2/N_1) - (\sigma^{(1)}/3)] \\ & = (\dot{\sigma}_{33}^{(2)}/2\mu_2) - (\nu_2/E_2)(\dot{\bar{\sigma}}_{11} + \dot{\sigma}_{22}^{(2)} + \dot{\sigma}_{33}^{(2)}) + \Lambda_2[\sigma_{33}^{(2)} - (\sigma^{(2)}/3)] \end{aligned} \tag{41}$$

$$(\dot{\sigma}_{32}^{(2)}/2\mu_1)(-N_2/N_1) + \Lambda_1\sigma_{32}^{(2)}(-N_2/N_1) = (\dot{\sigma}_{32}^{(2)}/2\mu_2) + \Lambda_2\sigma_{32}^{(2)}. \tag{42}$$

It is again understood that in the above equations, the relations (38) have been used in the expressions for $\sigma^{(1)}$, and Λ_α. The four eqns (39)–(42) can now be written in a compact matrix form as follows

$$\mathbf{C}\mathbf{W}_{,t} = \mathbf{R} \tag{43}$$

where $W_1 = \bar{\sigma}_{11}$, $W_2 = \sigma_{22}^{(2)}$, $W_3 = \sigma_{33}^{(2)}$, $W_4 = \sigma_{32}^{(2)}$, $\mathbf{C}$ is a four by four matrix of constant coefficients and $\mathbf{R}$ is a column matrix. The components of $\mathbf{C}$ and $\mathbf{R}$ are again in the Appendix.

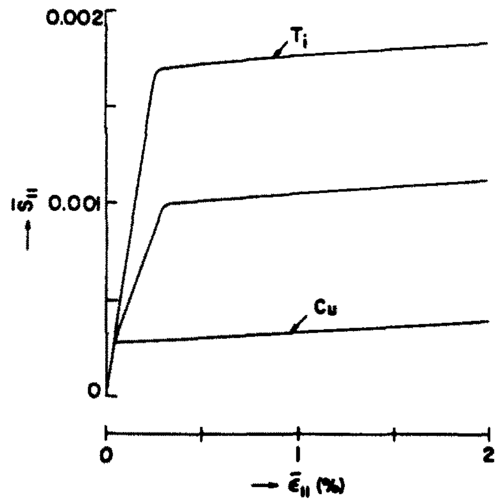

Fig. 5. Effective stress-strain curve for an elastic-viscoplastic composite thin rod made of titanium and copper and the stress-strain curves for a homogeneous thin rod made of (a) titanium (Ti), (b) copper (Cu). The strain rate in all case is $10^{-2}\,sec^{-1}$

The plastic work eqns (25) still apply, but the relations given in (37) need to be substituted in them. The dynamic equation of motion is given by:

$$\frac{\partial}{\partial x_1}\bar{\sigma}_{11} = \bar{\rho}\ddot{\bar{u}}_1. \tag{44}$$

It is therefore seen that the field equations describing the dynamic response of the composite rod are similar to those of the previous section except that eqn (43) is a four by four coupled system instead of the five by five one given in (33).

Equations (43) together with (25) are first used to obtain the equivalent stress-strain curve of the composite thin rod. The procedure is again the same as that of the previous section and the stress-strain curve is constructed for a strain rate of $\dot{\epsilon}_0 = 10^{-2}\,sec^{-1}$. The same notations are used as that of the previous section and Fig. 5 shows the equivalent stress-strain curve of the composite rod as compared to those of thin rods made of copper only and titanium only. It is seen that in this case of a thin rod, the uniaxial stress strain relations of copper and titanium are rather different and the equivalent model curve exhibits a double kink. This phenomenon is a

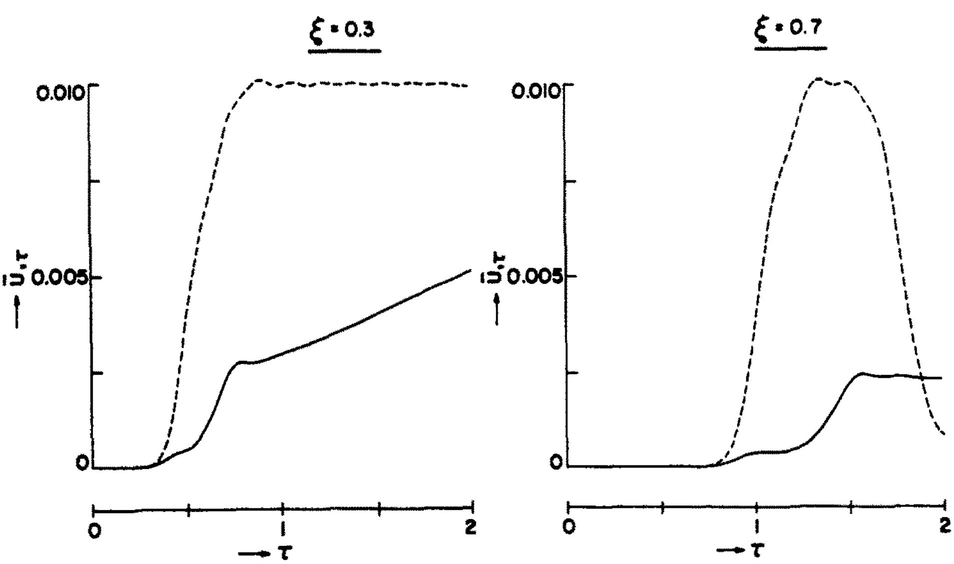

Fig. 6. The average velocity vs time at $\xi = 0.3$ and 0.7 in an elastic-viscoplastic composite thin rod (solid lines) and in a perfectly elastic thin composite rod (dashed lines). The velocity input is given by (35).

characteristic feature of the elastic–plastic behaviour of a composite structure. The first part of the equivalent curve indicates that both components are elastic, the middle part belongs to the situation when one component (copper) has yielded while the other one (titanium) is still elastic and the last part indicates that both components are plastic.

It should be noted that this double kink feature should theoretically appear also in Fig. 2 of the plane strain case, but since the behaviour of both copper and titanium are quite similar in that situation, presumably the double kink has been smeared into a single yield point.

The dynamic behaviour of a composite thin rod of length $H = l$ with the end $x_1 = H$ clamped, and the end $x_1 = 0$ subjected to a velocity input given by (36) is also investigated. The results are shown in Fig. 6 with all the parameters and notations being the same as those of the previous section. The response of the elastic–viscoplastic composite rod is compared with that of a composite rod made of titanium and copper but assuming this time that their behaviour is completely elastic. The effect of the viscoplastic mechanism is much more enhanced this time when compared to the results of the previous section.

REFERENCES

1. Z. Hashin, Theory of fiber reinforced materials. *NASA Rep. on Contract NASA* 1-8818 (1970).
2. G. W. Postma, Wave propagation in a stratified medium. *Geophysics* **20**, 780 (1955).
3. S. M. Rytov, Acoustical properties of a thinly laminated medium. *Soviet Phys. Acoust.* 2, 68 (1956).
4. S. R. Bodner and Y. Partom, Constitutive equations for elastic–viscoplastic strain-hardening material. *J. Appl. Mech.* **42**, 385 (1975).
5. J. Aboudi and Y. Benveniste, An average theory for the dynamic behaviour of a laminated elastic–viscoplastic work-hardening medium. Submitted for publication.
6. S. R. Bodner, private communication.

APPENDIX

The elements of the 9×9 matrix **A** and the column matrix **F** of eqn (27) are given by:

$$A_{11} = (N_1/2\mu_1) - (\nu_1 N_1/E_1) + (N_2/2\mu_2) - (\nu_2 N_2/E_2),$$

$$A_{14} = A_{16} = -\nu_1 N_1/E_1,\ A_{15} = A_{17} = -N_2\nu_2/E_2,$$

$$A_{22} = A_{33} = (N_1/2\mu_1) + (N_2/2\mu_2),\ A_{41} = A_{46} = A_{61} = A_{64} = -\nu_1/E_1,$$

$$A_{44} = A_{66} = (1/2\mu_1) - (\nu_1/E_1),\ A_{51} = A_{57} = A_{71} = A_{75} = -\nu_2/E_2,$$

$$A_{55} = A_{77} = (1/2\mu_2) - (\nu_2/E_2),\ A_{88} = (1/2\mu_1),\ A_{99} = (1/2\mu_2),$$

and all the other components of **A** are identically zero.

$$F_1 = -N_1\Lambda_1[\bar{\sigma}_{11} - (\sigma^{(1)}/3)] - N_2\Lambda_2[\bar{\sigma}_{11} - (\sigma^{(2)}/3)] + \frac{\partial}{\partial x_1}\dot{u}_1,$$

$$F_2 = -N_1\Lambda_1\bar{\sigma}_{12} - N_2\Lambda_2\bar{\sigma}_{12} + \left[\left(\frac{\partial}{\partial x_2}\dot{u}_1 + \frac{\partial}{\partial x_1}\dot{u}_2\right)/2\right],$$

$$F_3 = -N_1\Lambda_1\bar{\sigma}_{13} - N_2\Lambda_2\bar{\sigma}_{13} + \left[\left(\frac{\partial}{\partial x_3}\dot{u}_1 + \frac{\partial}{\partial x_1}\dot{u}_3\right)/2\right],$$

$$F_4 = -\Lambda_1[\sigma_{22}^{(1)} - (\sigma^{(1)}/3)] + \frac{\partial}{\partial x_2}\dot{u}_2,$$

$$F_5 = -\Lambda_2[\sigma_{22}^{(2)} - \sigma^{(2)}/3] + \frac{\partial}{\partial x_2}\dot{u}_2,$$

$$F_6 = -\Lambda_1[\sigma_{33}^{(1)} - (\sigma^{(1)}/3)] + \frac{\partial}{\partial x_3}\dot{u}_3,$$

$$F_7 = -\Lambda_2[\sigma_{33}^{(2)} - (\sigma^{(2)}/3)] + \frac{\partial}{\partial x_3}\dot{u}_3,$$

$$F_8 = -\Lambda_1\sigma_{23}^{(1)} + \left[\left(\frac{\partial}{\partial x_2}\dot{u}_3 + \frac{\partial}{\partial x_3}\dot{u}_2)/2\right],$$

$$F_9 = -\Lambda_2\sigma_{23}^{(2)} + \left[\left(\frac{\partial}{\partial x_2}\dot{u}_3 + \frac{\partial}{\partial x_3}\dot{u}_2\right)/2\right].$$

The elements of the 5×5 matrix $\mathbf{A}^*$ of eqn (30) are given by:

$$
\begin{aligned}
&A^*_{11}=A_{11},\ A^*_{12}=A^*_{14}=-(\nu_1N_1/E_1),\ A^*_{13}=A^*_{15}=-(N_2\nu_2/E_2)\\
&A^*_{21}=A^*_{24}=-(\nu_1/E_1),\ A^*_{22}=(1/2\mu_1)-(\nu_1/E_1),\ A^*_{23}=A^*_{25}=0,\\
&A^*_{31}=A^*_{35}=-(\nu_2/E_2),\ A^*_{32}=A^*_{34}=0,\ A^*_{33}=(1/2\mu_2)-(\nu_2/E_2)\\
&A^*_{41}=A^*_{42}=-(\nu_1/E_1),\ A^*_{43}=A^*_{45}=0,\ A^*_{44}=(1/2\mu_1)-(\nu_1/E_1),\\
&A^*_{51}=A^*_{53}=-(\nu_2/E_2),\ A^*_{52}=A^*_{54}=0,\ A^*_{55}=(1/2\mu_2)-(\nu_2/E_2).
\end{aligned}
$$

The elements of the 5×5 matrix $\mathbf{B}$ and column matrix $\mathbf{Q}$ of eqn (33) are given by

$$
\begin{aligned}
&B_{11}=A_{11},\ B_{12}=B_{14}=-(N_1\nu_1/E_1),\ B_{13}=B_{15}=-(N_2\nu_2/E_2),\\
&B_{21}=B_{24}=-(\nu_1/E_1),\ B_{22}=(1/2\mu_1)-(\nu_1/E_1),\ B_{23}=B_{25}=0\\
&B_{31}=B_{35}=-(\nu_2/E_2),\ B_{32}=B_{34}=0,\ B_{33}=(1/2\mu_2)-(\nu_2/E_2)\\
&B_{41}=B_{42}=-(\nu_1/E_1),\ B_{43}=B_{45}=0,\ B_{44}=(1/2\mu_1)-(\nu_1/E_1)\\
&B_{51}=B_{53}=-(\nu_2/E_2),\ B_{52}=B_{54}=0,\ B_{55}=(1/2\mu_2)-(\nu_2/E_2),\\
&Q_1=-N_1\Lambda_1[\sigma_{11}-(\sigma^{(1)}/3)]-N_2\Lambda_2[\sigma_{11}-(\sigma^{(2)}/3)]+\frac{\partial}{\partial x_1}\dot{\bar{u}}_1,\\
&Q_2=-\Lambda_1[\sigma^{(1)}_{22}-(\sigma^{(1)}/3)],\ Q_3=-\Lambda_2[\sigma^{(2)}_{22}-(\sigma^{(2)}/3)],\\
&Q_4=-\Lambda_1[\sigma^{(1)}_{33}-(\sigma^{(1)}/3)],\ Q_5=-\Lambda_2[\sigma^{(2)}_{33}-(\sigma^{(2)}/3)].
\end{aligned}
$$

The elements of the 4×4 matrix $\mathbf{C}$ and column matrix $\mathbf{R}$ of eqn (43) are given by

$$
\begin{aligned}
&C_{11}=A_{11},\ C_{12}=(\nu_1N_2/E_1)-(N_2\nu_2/E_2),\\
&C_{13}=(\nu_1N_1/E_1)-(N_2\nu_2/E_2),\ C_{14}=0\\
&C_{21}=-(\nu_1/E_1)+(\nu_2/E_2),\ C_{22}=(\nu_1N_2/E_1N_1)-(N_2/2\mu_1N_1)+(\nu_2/E_2)-(1/2\mu_2),\\
&C_{23}=(N_2\nu_1/N_1E_1)+(\nu_2/E_2),\ C_{24}=0\\
&C_{31}=-(\nu_1/E_1)+(\nu_2/E_2),\ C_{32}=(\nu_1N_2/E_1N_1)+(\nu_2/E_2),\\
&C_{33}=-(N_2/N_12\mu_1)+(N_2\nu_1/N_1E_1)+(\nu_2/E_2)-(1/2\mu_2),\\
&C_{34}=0,\ C_{41}=C_{42}=C_{43}=0,\ C_{44}=(-N_2/2\mu_1N_1)-(1/2\mu_2).\\
&R_1=-N_1\Lambda_1[\bar{\sigma}_{11}-(\sigma^{(1)}/3)]-N_2\Lambda_2(\bar{\sigma}_{11}-(\sigma^{(2)}/3)]+\frac{\partial}{\partial x_1}\dot{\bar{u}}_1,\\
&R_2=\Lambda_2[\sigma^{(2)}_{22}-(\sigma^{(2)}/3)]-\Lambda_1[-(\sigma^{(2)}_{22}N_2/N_1)-(\sigma^{(1)}/3)],\\
&R_3=\Lambda_2[\sigma^{(2)}_{33}-(\sigma^{(2)}/3)]-\Lambda_1[-(\sigma^{(2)}_{33}N_2/N_1)-(\sigma^{(1)}/3)],\\
&R_4=\Lambda_2\sigma^{(12)}_{23}+\Lambda_1(\sigma^{(2)}_{23}N_2/N_1).
\end{aligned}
$$